Commun. math. Phys. 4, 1—7 (1967)

Range of Forces and Broken Symmetries in Many-Body Systems*

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Received July 7, 1966

Abstract. It is argued that for a many-body system with short range forces the commutators between local operators at different times will be fast decreasing for large spatial separations.

This allows the adaptation of many discussions in relativistic field theories to the case of a many-body system with short range forces. In particular one has the result that a spontaneous breakdown in symmetry implies the existence of excitations of arbitrarily small energy. However this result has essentially only one application: We know that the Galilei invariance is always broken (in a medium of finite density). Therefore one concludes that in a many-body system with short range forces there can never be an energy gap.

I. Introduction

In the general framework of relativistic field theories[1] the postulate of commutativity of local observables for space-like separations plays a very fundamental role. A recent application of this postulate was a proof of Goldstone's theorem [3] which did not involve the existence of covariant field operators [4], [5]. In this connection we were led to search for the behaviour of commutators (anti-commutators) of local quantities at different times in a non-relativistic many-body system.

There one can safely start from canonical equal time (anti) commutation relations and through the equations of motion determine *in principle* the behaviour of (anti) commutators for different times. In practice of course we are not able to solve the equations of motion except in the two extreme cases of weak and strong coupling. The analysis of those two limiting situations in Section II gives us confidence that in general the rate of decrease of the commutator for fixed time difference and large spatial separations will be closely related to the rate of decrease of the interparticle potential.

Since the discussion of Goldstone's theorem in [4] could have been made by replacing the postulate of strict local-commutativity by the weakened assumption

$$\lim_{|\mathrm{x}|\to\infty} \|[A(\mathrm{x}, t), B]\| \; |\mathrm{x}|^n = 0 \tag{1}$$

* Supported by the National Science Foundation.

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[1] See for example [1], [2].

with $n = 2$, it could be expected that for a potential of sufficiently short range the results of [4] would be valid in a many-body system. This would mean that in a many-body system with short range forces and an energy gap any automorphism associated in the sense of [4] with a locally conserved current is unitarily implementable.

In Sections III and IV we show that due to the "breakdown" of Galilei invariance an energy gap in the density excitation spectrum is incompatible with short range forces. This prevents any further application of the results of [4].

The excitations of zero energy which are always present in a theory with Galilei invariant forces of short range are of the phonon type, i.e. they are connected to the ground state by the matter density operators. The quasi particle states in the BCS-theory of superconductivity are orthogonal to those states and could still have an energy gap. However, the gap can only be "absolute" due to the presence of the long range Coulomb forces which provide a mechanism for pushing up the energy of the Goldstone excitation to a finite value, the plasmon energy.

II. Commutators and range of forces

We consider the Hamiltonian

$$H = H_K + V \tag{2}$$

$$H_K = \frac{1}{2m} \int \nabla \psi^+(\mathrm{x}) \, \nabla \psi(\mathrm{x}) \, d^3x \tag{3}$$

$$V = \frac{1}{2} \int \psi^+(\mathrm{x}) \, \psi^+(\mathrm{y}) \, \psi(x) \, \psi(\mathrm{y}) \, V(\mathrm{x} - \mathrm{y}) \, d^3x \, d^3y \tag{4}$$

where ψ^+, ψ is a system of local creation and destruction operators, satisfying the canonical commutation or anticommutation relations and V is the interparticle potential.

In order to develop some feeling for the connection that ought to exist between the rate of decrease of the potential and that of the commutators at different times we consider the two limiting cases

a) $V = 0$

b) $H_K = 0 \, (m \to \infty)$.

For the case a), the free system, it is obvious that the (anti) commutator between

$$\psi_f(\mathrm{x}, t) = \int f(\mathrm{x}', t') \, \psi(\mathrm{x}' + \mathrm{x}, t' + t) \, d^3x' \, dt \tag{5}$$

and

$$\psi_g^+ = \int g(\mathrm{x}', t') \, \psi^+(\mathrm{x}', t') \, dx' \, dt' \tag{6}$$

(where f and g are smooth functions decreasing at infinity faster than any power) satisfies

$$\lim_{|\mathrm{x}|\to\infty} |\mathrm{x}|^n \, [\psi_f(\mathrm{x}, t), \psi_g^+]_\pm = \lim_{|\mathrm{x}|\to\infty} |\mathrm{x}|^n (2\pi) \int \tilde{f}\left(\mathbf{k}, \frac{k^2}{2m}\right) \times \\ \times \tilde{g}\left(-\mathbf{k}, \frac{-k^2}{2m}\right) e^{i\left(\mathbf{k}\,\mathrm{x} - \frac{k^2 t}{2m}\right)} d^3 k = 0 \tag{7}$$

for any positive n. Thus, in the absence of a potential, the commutator resp. anticommutator between two local operators decreases faster than any power for large spatial separations.

In the opposite case b) the exact solution of the equation of motion is

$$\psi(\mathrm{x}, t) = T(\mathrm{x}, t)\, \psi(\mathrm{x}, 0)$$

With

$$T(\mathrm{x}, t) = \exp\left(-i t \int V(\mathrm{x} - \mathrm{y})\, \psi^+(\mathrm{y}, 0)\, \psi(\mathrm{y}, 0)\, d^3 y\right). \tag{8}$$

The (anti) commutator becomes

$$[\psi(\mathrm{x}, t), \psi^+(\mathrm{y}, 0)]_\pm = \mp\left(e^{i t V(\mathrm{x}-\mathrm{y})} - 1\right) T(\mathrm{x}, t)\, \psi^+(\mathrm{y}, 0)\, \psi(\mathrm{x}, 0) \\ + \delta^3(\mathrm{x} - \mathrm{y})\, T(\mathrm{x}, t) \tag{9}$$

which means that for large spatial separation the (anti) commutator decreases like $t\, V(\mathrm{x} - \mathrm{y})$. Since the kinetic energy term by itself leads to a decrease faster than any power, it is very suggestive that the range of the commutator is in general controlled by the range of the potential through a relation similar to (9), namely

$$\|[A(\mathrm{x}, t), B(\mathrm{y}, 0)]\| < c\, |t\, V(\mathrm{x} - \mathrm{y})|, \tag{10}$$

if A and B are two bounded local observables and $|\mathrm{x} - \mathrm{y}|$ is large.

We cannot at present elaborate further on this point but will take for granted that (Eq. 1) will be satisfied for some n which is determined by the rate of decrease of the potential.

III. Breakdown of Galilei invariance

The reason that the results of [4] are not useful in many-body systems lies in the incompatibility between the assumptions of short-range forces and energy gap.

This incompatibility can be seen as follows: The non-relativistic many-body system has a group of algebraic automorphisms corresponding to Galilei transformations. In terms of the basic destruction operators they are given by

$$\psi(\mathrm{x}, t) \to \psi(\mathrm{x} - \mathrm{v}t, t) \exp i\left[m \mathrm{v}\,\mathrm{x} + \frac{m v^2}{2} t\right]. \tag{11}$$

This symmetry must be obviously broken since the ground state is in a preferred Galilei frame. The transformation (11) produces an infinite change in an infinitely extended system and can therefore not be unitarily implementable. As a consequence of the main theorem in [4] we conclude that either (1) does not hold (long range forces) or three is no energy gap.

More explicitly, if ϱ is the particle density, j the particle current, Ω the ground state of the system and $f_R(\mathrm{x})$ a function which has the value 1 for $|\mathbf{x}| < R$ and vanishes for $|\mathrm{x}| \gg R$ then the equal time commutation relations tell us that

$$\lim_{R\to\infty} \int \langle\Omega| \, [\varrho(\mathrm{x}), j_k(0)] \, |\Omega\rangle \, x_l f_R(\mathrm{x}) \, d^3x = i\langle\Omega| \, \varrho(0) \, |\Omega\rangle \delta_{kl} \neq 0 \, . \quad (12)$$

Note that

$$G_l^{(R)} = \int x_l \varrho(\mathrm{x}) \, f_R(\mathrm{x}) \, d^3x \quad (13)$$

becomes the generator of a Galilei transformation in the limit $R \to \infty$. Therefore we may expect that

$$\lim_{R\to\infty} \left\langle \Omega \middle| \left[\frac{\partial G_l^{(R)}}{\partial t}, B\right] \middle| \Omega \right\rangle = 0 \, , \quad (14)$$

if B is sufficiently local. Indeed the continuity equation

$$\frac{\partial \varrho}{\partial t} + \mathrm{div}\,\mathbf{j} = 0 \quad (15)$$

gives us

$$\left\langle \Omega \middle| \left[\frac{\partial G_l^{(R)}}{\partial t}, B\right] \middle| \Omega \right\rangle = \int \langle\Omega| \, [j_l(\mathrm{x}, 0), B] \, |\Omega\rangle \, f_R(\mathrm{x}) \, d^3x + \\ + \int \langle\Omega| \, [j_m(\mathrm{x}, 0), B] \, |\Omega\rangle \, x_l \frac{\partial}{\partial x_m} f_R(\mathrm{x}) \, {}^3x \, . \quad (16)$$

The second term on the right hand side of (16) vanishes in the limit $R \to \infty$ if B is quasilocal of order 3 because $\frac{\partial f_R}{\partial x_l} = 0$ for $|\mathbf{x}| < R$. Concerning the first term we observe that as a consequence of Galilei invariance the particle current j is also the momentum density[2] (with $m = 1$).

Therefore (for quasilocal B of order 3) the first term has the limit $\langle\Omega[P_l, B]\Omega\rangle$ which vanishes since Ω has linear momentum zero. This concludes the proof of (14) for any quasilocal of order 3 or better. If there is an energy gap and if the forces are of short range then, using the estimate (10), one easily applies the technique of ref. [4] to conclude that (14) implies also

$$\lim_{R\to\infty} \langle\Omega[G_l^R, j_k(0)]\Omega\rangle = 0$$

which is a contradiction to (12).

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[2] As easily seen from the structure relation

$$\frac{\partial}{\partial t} \int \varrho(\mathrm{x}) x_l d^3x \, d^3x = \int j_l(\mathrm{x}) \, d^3x = i[H, G_l] = P_l \, .$$

We have thus established that since in a many-body system there is always a spontaneously broken symmetry, the Galilei invariance, there can be no energy gap in the case of short range forces.

IV. Direct connection between spectrum and range of forces

In the previous section we assumed a rather loose connection between the range of forces and the behaviour of commutators which gave us some information about the spectrum. We shall approach this problem here in a more direct way by employing the method of sum rules, obtaining more quantitative results than in the previous section.

Many-body physicists have been using similar sum-rules [6] and mainly through them have understood for quite some time that in the presence of Coulomb forces a "Goldstonian" zero energy excitation can acquire a finite energy, becoming a plasmon [7], [8], [9].

We now prove the following: In a many-body system with a translationally invariant ground state, and a two body potential satisfying $\lim_{r\to\infty} r^{(1+\varepsilon)} V(r) = 0$, for some $\varepsilon > 0$, there is no energy gap.

Proof — Using the continuity equation (15) one arrives by a straightforward application of equal time commutation relations at the following sum rule

$$\tilde{F}(\mathbf{p}) = \int_0^\infty \omega \, d\mu_p(\omega) = \langle\Omega|\, \varrho(0)\, |\Omega\rangle p^2 \,, \tag{17}$$

where $\tilde{F}(\mathbf{p})$ is the Fourier transform of

$$F(\mathrm{x}) = \left(\Omega\middle|\left[\frac{\partial\varrho}{\partial t}(\mathrm{x},0),\varrho(0)\right]\middle|\Omega\right) \tag{18}$$

and $d\mu_p(\omega)$ a positive measure in ω.

Taking now the momentum conservation law

$$\frac{\partial}{\partial t} j_i(\mathrm{x},t) = \frac{\partial S_{ik}}{\partial x_k} - \psi^+(\mathrm{x},t)\int \nabla_i V(\mathrm{x}-\mathrm{y})\,\varrho(\mathrm{y},t)\,\psi(\mathrm{x},t) \tag{19}$$

with

$$\begin{aligned} S_{ik} = -\frac{1}{2}[\nabla_i\psi^+)(\nabla_k\psi) + (\nabla_k\psi^+)(\nabla_i\psi)] + \frac{\delta_{ik}}{4}[(\nabla^2\psi^+)\psi + \\ + \psi^+(\nabla^2\psi + 2(\nabla_l\psi^+)(\nabla_l\psi)] \end{aligned} \tag{20}$$

we find, using equal time commutation relations,

$$\begin{aligned} \left(\Omega\middle|\left[\frac{\partial}{\partial t} j_i(\mathrm{x},0), j_l(0)\right]\middle|\Omega\right) &= \left(\Omega\middle|\left[\frac{\partial S_{ik}}{\partial x_k}(\mathrm{x},0), j_l(0)\right]\middle|\Omega\right) + \\ &+ \frac{\delta^3(x)}{i}\nabla_l \int (\Omega|\,\varrho(0)\,\varrho(\mathrm{y},0)\,|\Omega)\,\nabla_i V(\mathrm{x}-\mathrm{y})\,d^3y - \\ &- \frac{1}{i}(\Omega|\,\varrho(0)\,\varrho(\mathrm{x},0)\,|\Omega)\,\nabla_l\nabla_i\, V(\mathrm{x}) \end{aligned} \tag{21}$$

and using (15) once more we find

$$\int_0^\infty \omega^3 \, d\,\mu_p(\omega) = 0\,(p^2) \quad \text{if} \quad \lim_{r\to\infty} r^{1+\varepsilon}\,V(r) = 0\,. \tag{22}$$

Comparing (17) and (22) one concludes:

(i) there can be no energy gap

(ii) the weight of $d\,\mu_p(\omega)$ gets entirely concentrated at the origin $\omega = 0$ for $\mathbf{p} = 0$:

$$\lim_{\mathbf{p}\to 0} \frac{\int_{a^2}^{\infty} d\,\mu_p(\omega)}{\int_0^{a^2} d\,\mu_p(\omega)} = 0\,. \tag{23}$$

V. Conclusions

We have shown that if the forces are faster decreasing than the Coulomb forces there are zero energy excitations which have been interpreted as "Goldstone-particles" associated with the breakdown of Galilei invariance.

Whether they are truly particle-like depends on whether their width (inverse lifetime) goes to zero faster than their energy for $\mathbf{p} \to 0$, and in contrast to the relativistic case (cf. [5]) this question cannot be settled on a general basis, as can be easily seen by considering the two cases of a free Fermi and a free Bose gas. In the first case we do not have particle-like excitations in the second we do.

Being present, those excitations can account for other spontaneously broken-symmetries. Whether a particular symmetry is spontaneously broken or not demands in the many body case considerably more information than the simple knowledge of the energy momentum spectrum.

The peculiar nature of Coulomb interactions as a means of preventing the appearance of "Goldstone-particles" has led authors [10], [11], [12], [13] to look for a similar mechanism in relativistic field theories by abandoning local commutativity in the presence of gauge fields. Those attempts bring to focus the point that although a theory should be ideally formulated in terms of observables whose commutation relations can be safely taken as local [2], those observables are precisely the gauge invariant quantities in terms of which the problem of breakdown of gauge invariance cannot be posed.

Acknowledgements. The author is much indebted to Professor R. HAAG for many stimulating discussions and suggestions and to Professor N. HUGENHOLTZ for his interest. The support of the National Science Foundation is gratefully acknowledged.

References

[1] STREATER, R. F., and A. S. WIGHTMAN: PCT, Spin statistics and all that. New York: W. A. Benjamin 1964.
[2] HAAG, R., and D. KASTLER: J. Math. Phys. **5**, 848 (1964).
[3] GOLDSTONE, J.: Nuovo cimento **19**, 154 (1961).
[4] KASTLER, D., D. ROBINSON, and A. SWIECA: Commun. Math. Phys. **2**, 108 (1966).
[5] EZAWA H. and J. A. SWIECA: Spontaneous breakdown of symmetries and zero mass states. (to be published)
[6] PINES, D.: Elementary excitations in solids. New York: W. A. Benjamin 1963.
[7] ANDERSON, P. W.: Phys. Rev. **112**, 1900 (1958).
[8] LANGE, R. V.: Phys. Rev. Letters **14**, **3** (1965).
[9] PINES, D., and P. NOZIÈRES: Theory of quantum liquids. New York: W. A. Benjamin 1966.
[10] ENGLERT, F., and R. BROUT: Phys. Rev. Letters **13**, 321 (1964).
[11] HIGGS, P. W.: Phys. Rev. Letters **13**, 508 (1964).
[12] — Spontaneous breakdown without massless Bosons (preprint).
[13] GURALNIK, G. S., C. R. HAGEN and T. W. KIBBLE: Phys. Rev. Letters **13**, 585 (1964).